Quantum Mechanics

The Two Slit Experiment

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A mathematical examination
of the two slit experiment.

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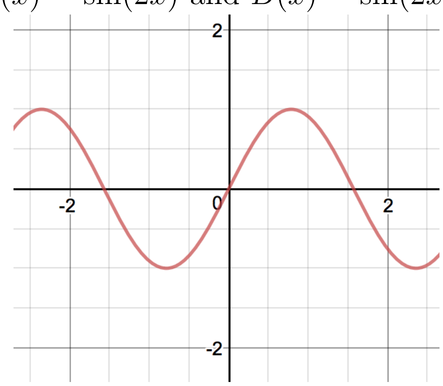
"*If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet*"
–Niels Bohr[7]

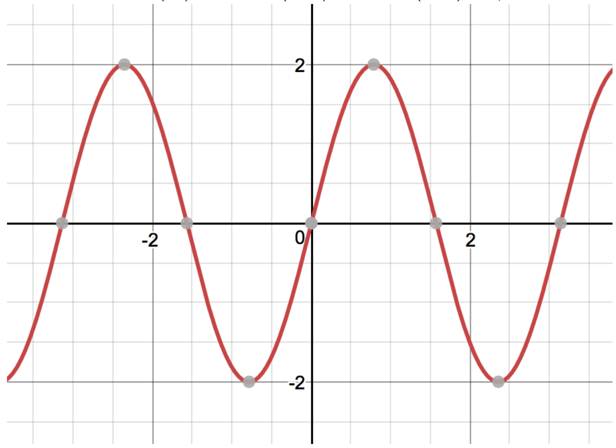
In this paper, I will examine the central mystery of quantum mechanics, as described by theoretical physicist Richard Feynman[5], the two slit experiment. Imagine there is a source of light shining against a screen with two slits. Assume the light is monochromatic, which is defined as light of a single wavelength. The light acts as ripples in a pond. As the light hits the screen, it squeezes through the two slits, and as a results each slit acts as a new source for the solid screen that is placed behind the screen with two slits. The light then spreads out and diffracts. As the light continues to spread out and the waves overlap, they interfere with each other. Where a crest hits a trough they cancel, and where a crest hits a crest they amplify. As a result, the solid screen ends with an interference pattern on it: a series of light and dark fringes. This occurs as a result of the waves either canceling or working together in phase. We can examine this through a mathematical example; Let one wave be $A(x) = \sin(2x)$ and the other be $B(x) = \sin(2x)$.

Figure 1: Graph of $A(x) = \sin(2x)$ and $B(x) = \sin(2x)$. (courtesy of Desmos)



These two waves added is $A + B = 2\sin(2x)$, which is know as constructive interference. The waves create destructive interference when $A(x) = \sin(2x)$ and $B(x) = \sin(2x + \Pi)$. Summing these two wave we get $A + B = 0$, hence the destruction.

Figure 2: Graph of $A(x) = \sin(2x) + \sin(2x)$. (courtesy of Desmos)



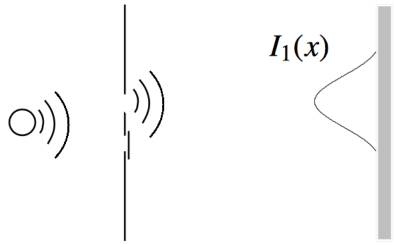
This is a property of light that has been known for 200 years, since the early 19th century. This was discovered by Thomas Young when he performed an experiment demonstrating interference from two closely spaced slits. This experiment deduced that light propagates as waves.[3]

The experimental outcome of sending a wave through slits can be modeled mathematical. We assume that waves are light waves with a wavelength of λ. The outcome can be denoted by a function $I_1(x)$, which is the square root of the value of the wave at a fixed point in time and space originating at the slit. Strictly operating with the electric field, where x is the fixed value of the wave, and t is time.Thus we can denote the intensity of the wave at a given time by the following function:

$$I_1(x) = |E_1(x,t)|^2 = E_1(x)^2[4]$$

The outcome is depicted in the figure below.

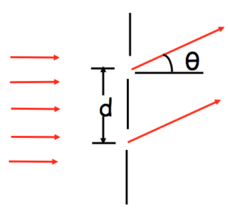
Figure 3: The result of wave traveling through the screen when only one slit is open. Curve $I_1(x)$ shows the intensity of the wave passing through the slit and reaching the screen at x.[4]



When both of the slits are open and the interference pattern occurs, the experimental outcome can be model by an intensity function which we will denote as $I_{12}(x)$. From the work of previously mentioned Thomas young we can model the intensity by the following:

The difference in lengths of the two paths that the waves will follow can be denoted by

Figure 4: waves passing through two slits[6]



$\Delta l = d \sin\theta$. Similarly to the example mentioned at the beginning of the paper about peaks and troughs of trigonometric functions, the intensity can be denoted for a trough can be denoted by:

$$d\sin\theta = (m \pm \frac{1}{2})\lambda, \quad m = 0, \pm 1, \pm 2, ...$$

and for the peak by:

$$d\sin\theta = (m\lambda), \quad m = 0, \pm 1, \pm 2, ...$$

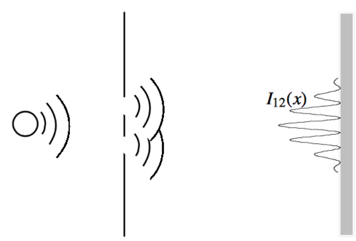
As mentioned previously, the intensity function is $I_1(x) = |E_1(x,t)|^2$, where $E_1 = E_0 \sin(t)$.

Thus for a second wave, $I_2(x) = |E_2(x,t)|^2$, where $E_2 = E_0 \sin(t+\phi)$ and $\phi = \frac{2\pi(d \sin\theta}{\lambda}$, that is θ giving position. The first step is to determine the resultant amplitude,E_R of the intensity functions. Let $I_{12}(x) = I_R$, which is our resultant intensity. We observe that we can write the intensity function as: $\frac{I_R}{I_n} = (\frac{E_R}{E_n})^2$, where n denotes the intensity of a particular wave.
Recall the trigonometric identity: $\sin x + \sin y = 2\cos[\frac{(a-b)}{2}]\sin[\frac{a+b}{2}]$
Using this identity we can see that,

$$\begin{aligned} E_1(x,t) + E_2(x,t) &= E_0[\sin(t) + \sin(t+\phi)] \\ &= 2E_0 \cos(\frac{\phi}{2})\cos(t+\frac{\phi}{2}) \end{aligned}$$

Note that the resultant amplitude is given by:$E_R = 2E_0\cos(\frac{\phi}{2})$. We can now observe that the resultant intensity is $I_{12}(x) = 4I_0\cos^2(\frac{\phi}{2})$, where phase difference between the two waves is $\phi = 2\pi\frac{\Delta l}{\lambda} = 2\pi\frac{d\sin\theta}{\lambda}$. The results are shown in the following figure.

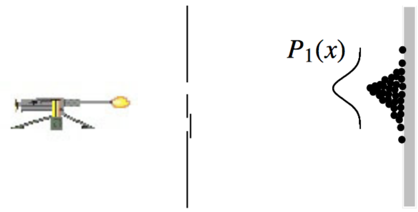
Figure 5: The result of the waves traveling through the screen when both slits are open. Curve $I_{12}(x)$ shows the intensity of the waves passing through the slit and reaching the screen at x.[4]



Suppose this experiment is repeated, not with waves but with particles, for example a grain of sand. Imagine that the experiment was rotated 90 degrees and the sand was poured at a screen with two slits vertically downward. Each particle goes through one slit or the other. As a result two piles of sand would be observed on the opposite side of the screen, aligned with where the slits in the screen are located. Since two peaks are observed, thus implying the coherence with the properties of particles. The experimental outcome can be modeled by the following probability function:

$$P_1(x)\delta x = \text{probability of sand landing in the range}(x, x+\delta)$$

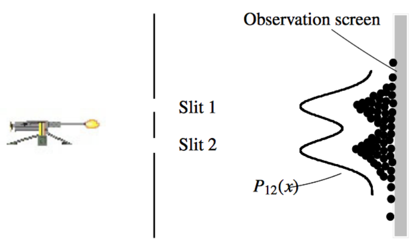
Figure 6: The result of sand poured through the screen when only one slit is open. Curve $P_1(x)$ shows the probability density of the sand passing through the slit.[4]



When both of the slits are open the experimental outcome can be modeled by the sum of the probability densities for each open slit and is denoted by the following:

$$P_{12} = P_1(x) + P_2(x)$$

Figure 7: The result of the sand poured through the screen when both slits are open. Curve $P_{12}(x)$ shows the probability density of the sand passing through each slit.[4]



Now suppose this experiment is attempted with electrons. Imagine a machine that will fire a stream of electrons, however, one of the slits in the screen is blocked off. After the atoms are shot and it is measured to which location they travel after passing through the slit, it is observed that some do not pass. The electrons that do pass will follow a general form appearing on the second screen. This will mimic the form that the sand produced but with only one slit. Thus we are able to denote the number of times the electron is observed on the screen as δx with a range of x to $x + \delta x$. The first mystery occurs when the second slit is unblocked. If the atoms are once again shot, an interference pattern similar to light is observed. Rather than having two bands of spots where the atoms have gone through the two slits, the atoms begin to behave like waves, and produce interference patterns.

Now suppose the atoms are sent through the two slits not all at once, but one at a time. Again, electrons may not make it through the first screen, but the electrons that do will reach the second screen, and thus produce some sort of pattern. After sending each electron through one at a time, gradually the same pattern will appear. Each atom by itself is contributing to the overall wave-like behavior in the interference pattern. The electron is a tiny localized particle and after shooting it at the screen it can be seen that the atom is a particular point; it has not spread itself apart, as an electron is localized. However, the electron is aware there are two slits as an interference pattern has occurred.[2]

In order to understand how the atom is behaving, the experiment is modified so that it can be observed how the atom behaves and which slit it passes through. In order to execute this, a detector is placed just above the upper slit of the screen and flashes or beeps to notify if it senses an atom has gone through the top slit. This detector is known as the Feynman light microscreope.Then again the atoms are fired through one at a time. The result observed is that 50% of the time the detector will beep and the other 50% it does not. The interference results are not the same as before; instead of an interference pattern there are two even collections of atoms distributed on the solid screen similar to the peaks of sand. As a last attempt to try to understand this bizarre occurrence, the experiment is once again modified. In an attempt to fool the atoms, the detector will be left in the same place, however, it will be turned off. After implementing the experiment with this modification it is observed the atoms return to an interference pattern.[2]

When the electron is shot with both slits open we observed that $P_{12} \neq P_1(x) + P_2(x)$. The result is actually of the form:

$$P_{12} = P_1(x) + P_2(x) + 2\sqrt{P_1(x)P_2(x)}\text{cos}\delta$$

The wavelength can be determined from the separation between the maxima of the interference pattern, thus $\delta = 2\pi d\text{sin}\theta/\lambda$, where $\lambda = h/p$, where h is heigh and p is the momentum of the electron. The waves can be represented using the de Broglie equation and denoted by:

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)[8]$$

. The amplitude of a wave passing through each slit at a particular point in time is given by $\Psi_1(x,t)$ and $\Psi_2(x,t)$. To make sense of $\Psi(x,t)$, we use the fact that $P_{12}\delta x$ is the probability of electron detection in the range $[x, x + \lambda x]$ we conjecture that

$$|\Psi(x,t)|^2\delta x \text{ is the probability of observing an electron in}[x, x + \delta x][1]$$

. This theory was originally proposed by Max Born and is known as probability amplitude.[1] We can also note that if the electron is passing through the slit with no detection system than the probability of the electron passing through the slit can be denoted by $P = |\Psi_1 + \Psi_2|^2$. However, when the electron is observed by a detector the probabilistic model is denoted by $P = P_1 + P_2$

$$\Psi = \sum_n A_n e^{i(p_n x - \omega_n t)} \lambda = \frac{h}{\gamma m_v}$$

This phenomenon defies human intuition. Mathematical concepts help humans understand the mystery of the two slit experiment and provide a fundamental base to quantum mechanics. Although conceptually, quantum mechanics is extremely difficult to grasp, the mathematics of quantum mechanics is predictive and aids the explanation of the subatomic world.

References

[1] Arie Caticha. Consistency, amplitudes, and probabilities in quantum theory. *Physical Review A (Atomic, Molecular, and Optical Physics)*, 57(3):1572–1582, March 1998.

[2] N. L. Chuprikov. Two-slit experiment and wave-particle duality. *Publication: Foundations of Probability and Physics - 6. AIP Conference Proceedings*, 1424(1):40–44, 03 2012.

[3] Physics Classroom. Propogation of an electromagnetic wave.

[4] J D Cresser. Quantum physics notes. *Department of Physics, Macquarie University*, 1:27–37, 2011. MR0671473, Zbl 0577.53007.

[5] Partha Ghose. The central mystery of quantum mechanics. *ARXIV*, 06 2009.

[6] McMaster. Interference of light ii. 2017.

[7] Brainy Quote. Quotes, 2018.

[8] P. Xu and L.-E Qiang. De Broglie relations, Gravitational time dilation and weak equivalence principle. *ArXiv e-prints*, 1, 2018.